

Products of free random variables and k -divisible non-crossing partitions

Octavio Arizmendi* and Carlos Vargas†
 Universität des Saarlandes, FR 6.1–Mathematik,
 66123 Saarbrücken, Germany

February 28, 2012

Abstract

We derive a formula for the moments and the free cumulants of the multiplication of k free random variables in terms of k -equal and k -divisible non-crossing partitions. This leads to a new simple proof for the bounds of the right-edge of the support of the free multiplicative convolution $\mu^{\boxtimes k}$, given by Kargin in [5], which show that the support grows at most linearly with k . Moreover, this combinatorial approach generalizes the results of Kargin since we do not require the convolved measures to be identical. We also give further applications, such as a new proof of the limit theorem of Sakuma and Yoshida [11].

1 Introduction and statement of results

Until recently, k -divisible non-crossing partitions have been overlooked in free probability and have barely appeared in the literature. However, their structure is very rich and there are, for instance, quite natural bijections between k -divisible non-crossing partitions and $(k+1)$ -equal partitions which preserve a lot of structure, (see e.g. [2]). Furthermore, as noticed in [1], the moments of $\mu^{\boxtimes k}$ can be computed using k -divisible non-crossing partitions.

In this paper we exploit the fact that k -divisible and k -equal partitions are linked, by the Kreweras complement, to partitions which are involved in the calculation of moments and free cumulants of the product of k free random variables. For details on free cumulants and their relevance in free probability, see [10].

Given $a, b \in \mathcal{A}$ free random variables, with free cumulants $\kappa_n(a)$ and $\kappa_n(b)$, respectively, one can calculate the free cumulants of ab by

$$\kappa_n(ab) = \sum_{\pi \in NC(n)} \kappa_\pi(a) \kappa_{Kr(\pi)}(b), \quad (1)$$

where $Kr(\pi)$ is the Kreweras complement of the non-crossing partition π . In particular, we are able to compute the free cumulants of the free multiplicative convolution of two compactly supported probability measures μ, ν , such that $Supp(\mu) \subseteq [0, \infty)$ by

$$\kappa_n(\mu \boxtimes \nu) = \sum_{\pi \in NC(n)} \kappa_\pi(\mu) \kappa_{Kr(\pi)}(\nu). \quad (2)$$

In principle, this formula could be inductively used to provide the free cumulants and moments of the convolutions of k (not necessarily equal) positive probability measures. This approach, however, prevents us from noticing a deeper combinatorial structure behind such products of free random variables.

Our fundamental observation is that, when π and $Kr(\pi)$ are drawn together, the partition $\pi \cup Kr(\pi) \in NC(2n)$ is exactly the Kreweras complement of a 2-equal partition (i.e. a non-crossing pairing). Furthermore, one can show using the previous correspondence that Equation (1) may be rewritten as

$$\kappa_n(ab) = \sum_{\pi \in NC_2(n)} \kappa_{Kr(\pi)}(a, b, \dots, a, b), \quad (3)$$

*Supported by Deutsche Forschungsgemeinschaft (DFG), Project SP419/8-1. E-mail: arizmendi@math.uni-sb.de

†Supported by Mexican National Council of Science and Technology (CONACYT) ref. 214839/310129. E-mail: carlos@math.uni-sb.de

where $NC_2(n)$ denotes the 2-equal partitions of $[2n]$.

Since 2-equal partitions explain the free convolution of two variables, it is natural to try to describe the product of k free variables in terms of k -equal partitions.

The main result of this work is the following.

Theorem 1. *Let $a_1, \dots, a_k \in (\mathcal{A}, \tau)$ be free random variables. Then the free cumulants and the moments of $a := a_1 \dots a_k$ are given by*

$$\kappa_n(a) = \sum_{\pi \in NC_k(n)} \kappa_{Kr(\pi)}(a_1, \dots, a_k), \quad (4)$$

$$\tau(a^n) = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(a_1, \dots, a_k), \quad (5)$$

where $NC_k(n)$ and $NC^k(n)$ denote, respectively, the k -equal and k -divisible partitions of $[kn]$.

The main application of our formulas is a new proof of the fact, first proved by Kargin [5], that for positive measures centered at 1, the support of the free multiplicative convolution $\mu^{\boxtimes k}$ grows at most linearly. Moreover, our approach enables us to generalize to the case $\mu_1 \boxtimes \dots \boxtimes \mu_k$, as follows.

Theorem 2. *There exists a universal constant $C > 0$ such that for all k and any μ_1, \dots, μ_k probability measures supported on $[0, L]$, satisfying $E(\mu_i) = 1$ and $\text{Var}(\mu_i) \geq \sigma^2$, for $i = 1, \dots, k$, the supremum L_k of the support of the measure $\mu_1 \boxtimes \dots \boxtimes \mu_k$ satisfies*

$$\sigma^2 k \leq L_k < CLk.$$

In other words, for (not necessarily identically distributed) positive free random variables $(X_i)_{i \geq 1}$ such that $E(X_i) = 1$, $\text{Var}(X_i) \geq \sigma^2$ and $\|X_i\| \leq L$, $i \geq 1$, we have that

$$\limsup_{n \rightarrow \infty} n^{-1} \|X_1^{1/2} \dots X_{n-1}^{1/2} X_n X_{n-1}^{1/2} \dots X_1^{1/2}\| < CL$$

and

$$\liminf_{n \rightarrow \infty} n^{-1} \|X_1^{1/2} \dots X_{n-1}^{1/2} X_n X_{n-1}^{1/2} \dots X_1^{1/2}\| \geq \sigma^2.$$

Let us point out that for the case $\mu_1 = \dots = \mu_k$, the previous theorem can be proved as using the methods of [7]. However, the norm estimates given there are meant to address more general situations (where certain linear combinations of products are allowed) and hence, the constants obtained using these methods for our specific problem are far from optimal.

Finally, we shall mention, that when $\mu_1 = \dots = \mu_k$, Kargin proved in [6] that

$$\lim_{k \rightarrow \infty} k^{-1} L_k = e\sigma^2.$$

The paper is organized as follows: Section 2 includes preliminaries on non-crossing partitions and the basic definitions from free probability that will be required. Some enumerative aspects of k -divisible partitions and their Kreweras complement are included. In Section 3 we prove our main formulae for the products of free random variables. We also express our results in terms of free multiplicative convolutions of compactly supported positive probability measures. Our formulas are used in Section 4 to derive bounds for the support of free multiplicative convolutions. Section 5 gives further applications and examples of our formulas. We calculate the free cumulants of products of free Poissons and shifted semicirculars. We show that the n -th cumulant of $\mu^{\boxtimes k}$ eventually becomes positive as k grows. We also provide a new proof of the limit theorem of Sakuma and Yoshida [11].

Acknowledgments

The authors would like to thank Professor Vladislav Kargin for pointing out reference [6]. They are grateful to Professor Roland Speicher for valuable comments and discussions.

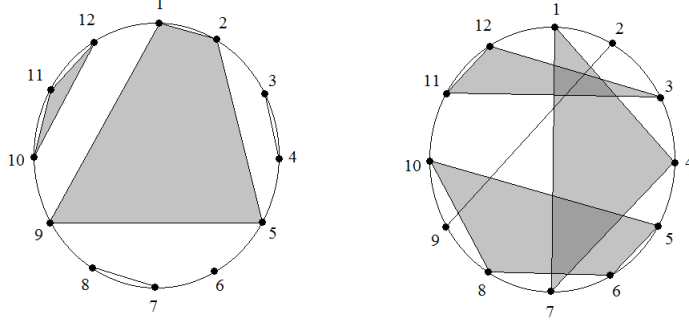


Figure 1: The non-crossing partition $\{\{1, 2, 5, 9\}, \{3, 4\}, \{6\}, \{7, 8\}, \{10, 11, 12\}\}$, and the crossing partition $\{\{1, 4, 7\}, \{2, 9\}, \{3, 11, 12\}, \{5, 6, 8, 10\}\}$ of the set $[12]$ in their circular representations.

2 Preliminaries

2.1 Non-crossing partitions

A *partition* of a finite, totally ordered set S is a decomposition $\pi = \{V_1, \dots, V_r\}$ of S into pairwise disjoint, non-empty subsets V_i , ($1 \leq i \leq r$), called *blocks*, such that $V_1 \cup V_2 \cdots \cup V_r = S$. The number of blocks of π is denoted by $|\pi|$.

A partition $\pi = \{V_1, \dots, V_r\}$ is called *non-crossing* if for every 4-tuple $a < b < c < d \in S$ such that $a, c \in V_i$ and $b, d \in V_j$, we have that $V_j = V_i$. We denote by $NC(S)$ the set of non-crossing partitions of S . For $S = [n] := \{1, 2, \dots, n\}$ we simply write $NC(n)$.

$NC(S)$ can be equipped with the partial order \preceq of reverse refinement ($\pi \preceq \sigma$ if and only if every block of π is completely contained in a block of σ). This turns $NC(S)$ into a lattice, in particular, we can talk about the least upper bound $\pi_1 \vee \pi_2 \in NC(S)$ of two non-crossing partitions $\pi_1, \pi_2 \in NC(S)$. We denote the minimum and maximum partitions of $NC(n)$ by $0_n, 1_n$, respectively. We will write

$$\rho_k^n := \{(1, \dots, k), (k+1, \dots, 2k), \dots, ((k-1)n+1, \dots, kn)\}$$

for the partition in $NC(kn)$ consisting of n consecutive blocks of size k .

Definition 3. We say that a non-crossing partition π is *k-divisible* if the sizes of all the blocks of π are multiples of k . If, furthermore, all the blocks are exactly of size k , we say that π is *k-equal*. A partition $\pi \in NC(nk)$ is called *k-preserving* if all its blocks contain numbers with the same congruence modulo k . A *k-preserving* partition $\pi \in NC(nk)$ is called *k-completing* if $\pi \vee \rho_k^n = 1_{nk}$.

Later we will see that these concepts are closely related. We denote the set of k -divisible non-crossing partitions of $[kn]$ by $NC^k(n)$ and the set of k -equal non-crossing partitions of $[kn]$ by $NC_k(n)$.

It is helpful to picture partitions via their circular representation: We think of $[n]$ as the clockwise labelling of the vertices of a regular n -gon. If we identify each block of $\pi \in NC(n)$ with the convex hull of its corresponding vertices, then we see that π is non-crossing precisely when its blocks are pairwise disjoint (that is, they don't cross).

It is well known that the number of non-crossing partitions is given by the Catalan numbers $C_n := \frac{1}{n+1} \binom{2n}{n}$. The following formula (proved by Kreweras [8]) counts the number of partitions of a given type.

Proposition 4. Let n be a positive integer and let $r_1, r_2, \dots, r_n \in \mathbb{N} \cup \{0\}$ be such that $r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC(n)$ with r_1 blocks with 1 element, r_2 blocks with 2 elements, \dots , r_n blocks with n elements equals

$$\frac{n!}{\left(n+1 - \sum_{i=1}^n r_i\right)! \prod_{i=1}^n r_i!}. \quad (6)$$

From the previous proposition one can easily count k -equal partitions.

Corollary 5.

$$|NC_k(n)| = \frac{\binom{kn}{n}}{(k-1)n+1}.$$

Finally, the number of k -divisible partitions was essentially given by Edelman [4].

Proposition 6.

$$|NC^k(n)| = \frac{\binom{(k+1)n}{n}}{kn+1}.$$

Out of two non-crossing partitions $\pi_1, \pi_2 \in NC(n)$ we can build the 2-preserving partition $\pi_1 \cup \pi_2$ by thinking $\pi_1 \in NC(\{1, 3, \dots, 2n-1\})$, $\pi_2 \in NC(\{2, 4, \dots, 2n\})$, and drawing them together. In general, the resulting partition may have crossings. Then for a given $\pi \in NC(n) \cong NC(\{1, 3, \dots, 2n-1\})$ we define its *Kreweras complement*

$$Kr(\pi) := \max\{\sigma \in NC(n) \cong NC(\{2, 4, \dots, 2n\}) : \pi \cup \sigma \in NC(2n)\}.$$

The Kreweras complement [8] satisfies many nice properties. The map $Kr : NC(n) \rightarrow NC(n)$ is an order reversing isomorphism. Furthermore, for all $\pi \in NC(n)$ we have that $|\pi| + |Kr(\pi)| = n+1$.

Similar to Proposition 4, one can count the number of partitions π , such that π and $Kr(\pi)$ have certain block structures. Let $(r_i)_{1 \leq i \leq n}$, $(b_j)_{1 \leq j \leq n}$ be tuples satisfying

$$1r_1 + 2r_2 + \dots + nr_n = n = 1b_1 + 2b_2 + \dots + nb_n, \quad (7)$$

$$|\pi| + |Kr(\pi)| = r_1 + \dots + r_n + b_1 + \dots + b_n = n+1. \quad (8)$$

Then the number of partitions such that π has r_i blocks of size i and $Kr(\pi)$ has b_j blocks of size j is given by the formula

$$n \frac{(|\pi| - 1)! (|Kr(\pi)| - 1)!}{r_1! \dots r_n! b_1! \dots b_n!}. \quad (9)$$

When π is k -equal, Equation (9), reduces to

$$k \frac{((k-1)n)!}{b_1! \dots b_n!}.$$

As a consequence, we can show that for large k , the Kreweras complements of k -equal partitions have “typically” small blocks. More precisely, for $n, k \geq 1$ let

$$NC(k, n)_{2,1} := \{\pi \in NC_k(n) : Kr(\pi) \text{ contains only pairings and singletons}\} \subseteq NC_k(n).$$

In this case, the only possibility is that $b_1 = n(k-2) + 2$, $b_2 = n-1$ and $b_i = 0$ for $i > 2$. Hence

$$|NC(k, n)_{2,1}| = k \frac{((k-1)n)!}{((n(k-2)+2)!(n-1)!)}. \quad (10)$$

An easy application of Stirling’s approximation formula shows that

$$\lim_{k \rightarrow \infty} \frac{|NC(k, n)_{2,1}|}{|NC_k(n)|} = 1. \quad (11)$$

2.2 Free Probability and Free Cumulants

A C^* -probability space is a pair (\mathcal{A}, τ) , where \mathcal{A} is a unital C^* -algebra and $\tau : \mathcal{A} \rightarrow \mathbb{C}$ is a positive unital linear functional.

The *free cumulant functionals* $(\kappa_n)_{n \geq 1}$, $\kappa_n : \mathcal{A}^n \rightarrow \mathbb{C}$ are defined inductively and indirectly by the moment-cumulant formula:

$$\tau(a_1 \dots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \dots, a_n), \quad (12)$$

where, for a family of functionals $(f_n)_{n \geq 1}$ and a partition $\pi \in NC(n)$, $\pi = \{V_1, \dots, V_i\}$, $V_j = \{m_1^{(j)}, \dots, m_{|V_j|}^{(j)}\} \subseteq [n]$, we define

$$f_\pi(a_1, \dots, a_n) := \prod_{j=1}^i f_{|V_j|}(a_{m_1^{(j)}} \dots a_{m_{|V_j|}^{(j)}}). \quad (13)$$

If we put $\tau_n(a_1, \dots, a_n) := \tau(a_1 \cdots a_n)$, Formula 12 may be inverted as follows:

$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \tau_\pi(a_1, \dots, a_n) \text{Mob}[\pi, 1_n], \quad (14)$$

where $\text{Mob} : NC(n) \times NC(n) \rightarrow \mathbb{C}$ is the Möbius function on $NC(n)$. For a detailed exposition we refer to [10, Lecture 11].

The main property of the free cumulants is that they characterize free independence. Here we will actually employ this characterization as the definition for freeness (for the equivalence with the usual definition see e.g. [10]).

The variables $a_1, \dots, a_k \in \mathcal{A}$ are *free* if and only if we have, for all $n \in \mathbb{N}$ and $1 \leq i_1, \dots, i_n \leq k$, that

$$\kappa_n(a_{i_1}, \dots, a_{i_n}) = 0,$$

unless $i_1 = i_2 = \dots = i_n$.

The main tool used to derive our formulae is the following, proved by Krawczyk and Speicher [9].

Proposition 7 (Formula for products as arguments). *Let (\mathcal{A}, τ) be a C^* -probability space and let $(\kappa_n)_{n \geq 1}$ be the corresponding free cumulants. Let $m, n \geq 1$, $1 \leq i(1) < i(2) < \dots < i(m) = n$ be given and consider the partition*

$$\hat{0}_m = \{\{1, \dots, i(1)\}, \dots, \{i(m-1) + 1, \dots, i(m)\}\} \in NC(n).$$

Consider random variables $a_1, \dots, a_n \in \mathcal{A}$. Then the following equation holds:

$$\kappa_m(a_1 \cdots a_{i(1)}, \dots, a_{i(m-1)+1} \cdots a_{i(m)}) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \hat{0}_m = 1_n}} \kappa_\pi(a_1, \dots, a_n). \quad (15)$$

It is well known that for a self-adjoint element $a \in \mathcal{A}$ there exist a unique compactly supported measure μ_a (its distribution) with the same moments as a , that is,

$$\int_{\mathbb{R}} x^k \mu_a(dx) = \tau(a^k), \quad \forall k \in \mathbb{N}.$$

In particular, if $a \in \mathcal{A}$ is positive (i.e. $a = bb^*$ for some $b \in \mathcal{A}$), the measure μ_a is supported in the positive half-line. Furthermore, given compactly supported probability measures μ_1, \dots, μ_k we can find free elements a_1, \dots, a_k in some C^* -probability space such that $\mu_{a_i} = \mu_i$.

For compactly supported measures on the positive half-line μ_1, \dots, μ_k , we can then consider positive elements $a_1, \dots, a_k \in \mathcal{A}$ such that $\mu_i = \mu_{a_i}$. Positive elements have unique positive square roots, so let b_1, \dots, b_k be the (necessarily free) positive elements such that $b_i^2 = a_i$.

Then, on one hand, the *free multiplicative convolution* is defined as the distribution $\mu_1 \boxtimes \dots \boxtimes \mu_k$ of the positive element $b := b_1 b_2 \cdots b_k b_k \cdots b_1$. It is not hard to see that for tracial C^* -probability spaces, the moments of b are exactly the moments of $a := a_1 \cdots a_k$. It will be convenient for us to work with a instead of b .

On the other hand, the *free additive convolution* is defined as the distribution $\mu_1 \boxplus \dots \boxplus \mu_k$ of the positive element $a_1 + a_2 + \dots + a_k$. It is easy to see that $\kappa_n(\mu_1 \boxplus \dots \boxplus \mu_k) = \kappa_n(\mu_1) + \dots + \kappa_n(\mu_k)$.

Finally, for a measure μ and $c > 0$ we denote by $D_c(\mu)$ the measure such that $D_c(\mu)(B) = \mu(\frac{1}{c}B)$ for all Borel sets B . In this case, $\kappa_n(D_c(\mu)) = c^n \kappa_n(\mu)$.

3 Main Formulas

The following characterization plays a central role in this work. The proof, elementary but cumbersome, can be found in the Appendix.

Proposition 8. *i) $\pi \in NC(kn)$ is k -preserving if and only if $\pi = Kr(\sigma)$ for some k -divisible partition $\sigma \in NC^k(n)$.*

ii) $\pi \in NC(kn)$ is k -completing if and only if $\pi = Kr(\sigma)$ for some k -equal partition $\sigma \in NC_k(n)$.

Remark 9. *In view of the previous characterization, for a k -divisible partition π , the Kreweras complement $Kr(\pi)$ may be divided into k partitions $\pi_1, \pi_2, \dots, \pi_k$, with π_j involving only numbers congruent to $j \bmod k$. In this case we will write $\pi_1 \cup \dots \cup \pi_k = Kr(\pi)$ for such decomposition.*

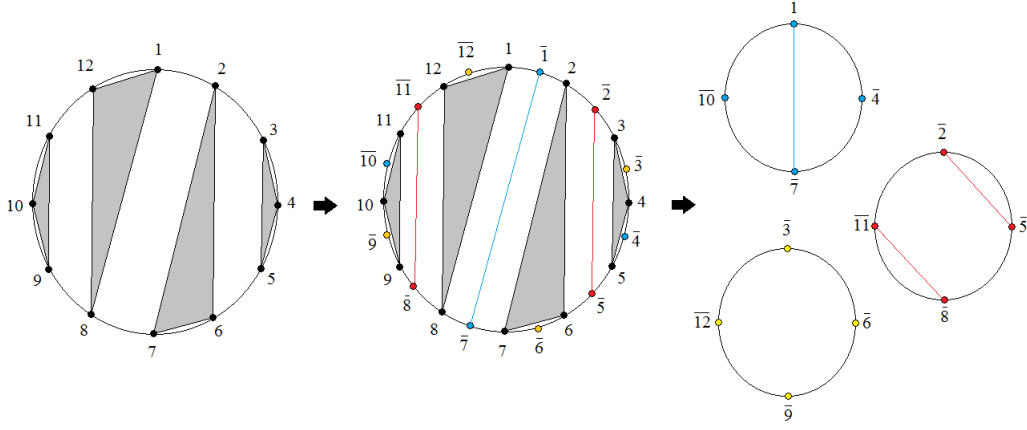


Figure 2: The 3-equal partition $\pi = \{\{1, 8, 12\}, \{2, 6, 7\}, \{3, 4, 5\}, \{9, 10, 11\}\}$ and its Kreweras complement $Kr(\pi) = \pi_1 \cup \pi_2 \cup \pi_3$, with $\pi_1 = \{\{1, 7\}, \{4\}, \{10\}\}$, $\pi_2 = \{\{2, 5\}, \{8, 11\}\}$ and $\pi_3 = \{\{3\}, \{6\}, \{9\}, \{12\}\}$.

We are ready to prove our main Theorem.

Proof of Theorem 1. By the formula for products as arguments, we have that

$$\kappa_n(a) = \prod_{\substack{\pi \in NC(kn) \\ \pi \vee \rho_n^k = 1_{nk}}} \kappa_\pi(a_1, \dots, a_n).$$

Since the random variables are free, the sum runs actually over k -preserving partitions (otherwise there would be a mixed cumulant). But then by Proposition 8 ii), the partitions involved in the sum are exactly the Kreweras complements of k -equal partitions, and the formula follows.

For the proof of (5), we use the moment-cumulant formula

$$\tau(a^n) = \sum_{\pi \in NC(kn)} \kappa_\pi(a_1, \dots, a_n).$$

Again, the elements involved are free, so only k -preserving partitions matter, and these are the Kreweras complements of k -divisible partitions by Proposition 8 i). Hence the result follows. \square

Remark 10. As pointed out in [3], Equation (1) is also satisfied when we replace the free cumulants by the Boolean cumulants. Therefore, Formula (4) holds as well for Boolean cumulants $(b_n)_{n \geq 1}$, namely, if $a := a_1 \cdots a_k$ is a product of free random variables, we have

$$b_n(a) = \sum_{\pi \in NC_k(n)} b_{Kr(\pi)}(a_1, \dots, a_k). \quad (16)$$

Corollary 11. Let μ_1, \dots, μ_k be probability measures with positive bounded support and let $\mu = \mu_1 \boxtimes \cdots \boxtimes \mu_k$. We may rewrite our formulas,

$$\kappa_n(\mu) = \sum_{\pi \in NC_k(n)} \kappa_{\pi_1}(\mu_1) \cdots \kappa_{\pi_k}(\mu_k), \quad (17)$$

$$m_n(\mu) = \sum_{\pi \in NC^k(n)} \kappa_{\pi_1}(\mu_1) \cdots \kappa_{\pi_k}(\mu_k), \quad (18)$$

where $\pi_1 \cup \cdots \cup \pi_k = Kr(\pi)$ is the decomposition described in Remark 9.

Remark 12. From Equation (18) it is easy to see that for compactly supported measures with mean 1, the variance is additive with respect to free multiplicative convolution, that is

$$Var(\mu_1 \boxtimes \cdots \boxtimes \mu_k) = \kappa_2(\mu_1 \boxtimes \cdots \boxtimes \mu_k) = \sum_{i=1}^k \kappa_2(\mu_i) = \sum_{i=1}^k Var(\mu_i).$$

4 Supports of free multiplicative convolutions

Our main result can be used to compute bounds for the supports of free multiplicative convolutions of positive measures. Our results in this section generalize those by Kargin in [5], where the case $\mu_1 = \dots = \mu_k$ was treated.

We recall that the cardinality of $NC(n)$ is given by the Catalan number C_n , which is easily bounded by 4^n . It is also known that the Möbius function $Mob : NC(n) \times NC(n) \rightarrow \mathbb{C}$ is bounded in absolute value by $C_{n-1} \leq 4^{n-1}$ (see [10, Prop 13.15]). Thus we are able to control the size of the free cumulants.

Lemma 13. *Let μ be a probability measure supported on $[0, L]$ with variance σ^2 , such that $E(\mu) = 1$. Then $\kappa_2(\mu) = \sigma^2 \leq L - 1$ and $|\kappa_n(\mu)| < (26L)^{n-1}$.*

Proof. It is easy to see that $L \geq 1$ and therefore

$$m_n^\mu = \int_0^L x^n d\mu(x) \leq \int_0^L L^{n-1} x d\mu(x) = L^{n-1}.$$

Then we have that $\kappa_1(\mu) = 1$, $0 < \kappa_2(\mu) \leq L - 1 < 26L$, and $|\kappa_3(\mu)| \leq L^2 + 3L + 1 < 26^2 L^2$, and for $n \geq 4$ we have

$$|\kappa_n(\mu)| = \sum_{\pi \in NC(n)} |m_\pi^\mu| |Mob[\pi, 1_n]| \leq \sum_{\pi \in NC(n)} L^{n-1} 4^{n-1} \leq 4^{2n-1} L^{n-1} < (26L)^{n-1},$$

since $4^7 < 26^3$. □

We easily see that the growth of the support is no less than linear.

Proposition 14. *Let μ_1, \dots, μ_k be compactly supported probability measures on \mathbb{R}^+ , satisfying $E(\mu_i) = 1$, $Var(\mu_i) = \sigma^2$, and let L_k be the supremum of the support of $\mu := \mu_1 \boxtimes \dots \boxtimes \mu_k$. Then $L_k \geq k\sigma^2 + 1$.*

Proof. It is clear that $E(\mu) = 1$, and hence by Remark 12 we know that $\kappa_2(\mu) = Var(\mu) = k\sigma^2$. By Lemma 13 we have that $\kappa_2(\mu) = k\sigma^2 \leq L_k - 1$. □

Now, we give an upper bound for the support of the free multiplicative convolution $\mu_1 \boxtimes \dots \boxtimes \mu_k$.

Proposition 15. *There exist a universal constant C , such that for all k and all μ_1, \dots, μ_k probability measures supported on $[0, L]$, satisfying $E(\mu_i) = 1$, $i = 1, \dots, k$, the measure $\mu := \mu_1 \boxtimes \dots \boxtimes \mu_k$ satisfies*

$$Supp(\mu) \subseteq [0, CL(k+1)].$$

In general, C may be taken $\leq 26e$. If the measures μ_i , $1 \leq i \leq k$ have non-negative free cumulants, C may be taken $\leq e$.

Proof. By Equation (18) we get

$$m_n(\mu) = \sum_{\pi \in NC^k(n)} \kappa_{\pi_1}(\mu_1) \dots \kappa_{\pi_k}(\mu_k) \tag{19}$$

Since a k -divisible partition $\pi \in NC^k(n)$ has at most n blocks, we know that

$$|\pi_1| + \dots + |\pi_k| = |Kr(\pi)| = kn + 1 - |\pi| \geq (k-1)n + 1.$$

Now, let $\tilde{L} = 26L$. By Lemma 13, we know that $\kappa_{\pi_i}(\mu_i) \leq (\tilde{L})^{n-|\pi_i|}$. Hence

$$\sum_{\pi \in NC^k(n)} \kappa_{\pi_1}(\mu_1) \dots \kappa_{\pi_k}(\mu_k) \leq \sum_{\pi \in NC^k(n)} (\tilde{L})^{kn - (|\pi_1| + \dots + |\pi_k|)} \tag{20}$$

$$\leq \sum_{\pi \in NC^k(n)} (\tilde{L})^n \tag{21}$$

$$= \frac{\binom{(k+1)n}{n}}{kn+1} (\tilde{L})^n \tag{22}$$

By taking the n -th root and the use of Stirling approximation formula, we obtain that

$$\limsup_{n \rightarrow \infty} (m_n(\mu))^{1/n} = \frac{(k+1)^{(k+1)}}{k^k} (\tilde{L}) \quad (23)$$

$$\leq (k+1)e\tilde{L}. \quad (24)$$

If μ has non-negative free cumulants we may replace \tilde{L} by L . \square

One may think from Propositions 14 and 15 that the measures $\nu_k := D_{1/k}(\mu^k)$ converge to a non-trivial limit. However, since $E[\nu_k] \rightarrow 0$ and $E[(\nu_k)^2] \rightarrow 0$, then $\nu_k \rightarrow \delta_0$.

5 More applications and examples

In this section we want to show some examples of how Theorem 1 may be used to calculate free cumulants.

Example 16. (*Product of free Poissons*) Theorem 1 takes a very easy form in the particular case $\mu_i = m$, where m is the Marchenko-Pastur distribution of parameter 1. Indeed, since $\kappa_n(m) = 1$, we get

$$\kappa_n(m^{\boxtimes k}) = \sum_{\pi \in NC^k(n)} 1 = \frac{\binom{kn}{n}}{(k-1)n+1},$$

and

$$m_n(m^{\boxtimes k}) = \sum_{\pi \in NC_k(n)} 1 = \frac{\binom{(k+1)n}{n}}{kn+1}.$$

Moreover, from the last equation one can easily calculate the supremum of the support $L_k = (k+1)^{k+1}/k^k$.

Example 17. (*Product of shifted semicirculars*) For $\sigma^2 \leq \frac{1}{4}$, let $\omega_+ := \omega_{1,\sigma^2}$ be the shifted Wigner distribution with mean 1 and variance σ^2 . The density of ω_+ is given by

$$\omega_{1,\sigma^2}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x-1)^2} \cdot 1_{[1-2\sigma, 1+2\sigma]}(x) dx,$$

and its free cumulants are $\kappa_1(\omega_+) = 1$, $\kappa_2(\omega_+) = \sigma^2$ and $\kappa_n(\omega_+) = 0$ for $n > 2$.

We want to calculate the free cumulants of $\omega_+^{\boxtimes k}$, $k \geq 2$. So let a_1, \dots, a_k be free random variables with distribution ω_+ . By Theorem 1, the free cumulants of $a := a_1 \cdots a_k$ are given by

$$\kappa_n(a) = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(a_1, \dots, a_k). \quad (25)$$

If $Kr(\pi)$ contains a block of size greater than 2, then $\kappa_{Kr(\pi)} = 0$. Hence the sum runs actually over $NC(k, n)_{2,1}$. Therefore each summand has the common contribution of $(\sigma^2)^{n-1}$ and by Equation (10) we know the number of summands. Then the free cumulants are

$$k \frac{((k-1)n)!(\sigma^2)^{n-1}}{(n-1)!((k-2)n+2)!}. \quad (26)$$

Note that in this example $m_n(a) \geq \kappa_n(a)$ (since all the free cumulants are positive) and $L = 1 + 2\sigma$. By Proposition 15 and an application of Stirling's formula to Equation (26) the supremum L_k of the support of $\omega_+^{\boxtimes k}$ satisfies

$$(k+1)e(1+2\sigma) \geq L_k = \limsup_{n \rightarrow \infty} (m_n(a))^{1/n} \geq \limsup_{n \rightarrow \infty} (\kappa_n(a))^{1/n} \geq (k-1)e\sigma^2.$$

Hence we obtain a better estimate than the rough bound provided by Proposition 14.

As another application of our main formula, we show that the free cumulants of $\mu^{\boxtimes k}$ become positive for large k . This is of some relevance if we recall that our estimates of the support of $\mu^{\boxtimes k}$ are better with the presence of non-negative free cumulants.

Theorem 18. *Let μ be a probability measure supported on $[0, L]$ with mean α and variance σ^2 . Then for each $n \geq 1$ there exist a constant N such that for all $k \geq N$, the first n free cumulants of $\mu^{\boxtimes k}$ are non-negative.*

Proof. Clearly it is enough to show that, for each $n \geq 1$, there exist N_0 such that the n -th free cumulant of $\mu^{\boxtimes k}$ is positive for all $k \geq N_0$.

Let $n > 1$ and $\tilde{\alpha} := \max\{\alpha^{n-1}, 1\}$. By the same arguments as in Lemma 13 one can show that $|\kappa_n(\mu)| \leq 16(L^n)$. Then by Theorem 1 we have

$$\begin{aligned} \kappa_n(\mu^{\boxtimes k}) &= \sum_{\pi \in NC(k, n)_{2,1}} \kappa_{Kr(\pi)}(\mu) + \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k, n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \\ &\geq \alpha^{(k-2)n+2} \left(\sum_{\pi \in NC(k, n)_{2,1}} \sigma^{2n-2} - \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k, n)_{2,1}}} (16L)^{n-1} \tilde{\alpha} \right) \\ &= \alpha^{(k-2)n+2} (|NC(k, n)_{2,1}| \sigma^{2n-2} - (|NC_k(n)| - |NC(k, n)_{2,1}|) (16L)^{n-1} \tilde{\alpha}). \end{aligned}$$

The factor $\alpha^{(k-2)n+2}$ is positive and the rest of the expression becomes positive for all k larger than some N_0 , since, by Equation (11), $|NC(k, n)_{2,1}|/|NC_k(n)| \rightarrow 1$ as $k \rightarrow \infty$. \square

It would be interesting to investigate whether or not all free cumulants become positive.

In a recent paper [11], Sakuma and Yoshida found a probability measure \mathfrak{h}_{σ^2} , arising from a limit theorem, which is infinitely divisible with respect to both multiplicative and additive free convolutions. Using our methods, we give another proof of this limit theorem. We restrict to the case $E(\mu) = 1$. The general case follows directly from this.

Proposition 19. *Let μ be a probability measure supported on $[0, L]$, with $E(\mu) = 1$ and $Var(\mu) = \sigma^2$, then*

$$\lim_{k \rightarrow \infty} D_{1/k} \left((\mu^{\boxtimes k})^{\boxplus k} \right) = \mathfrak{h}_{\sigma^2}$$

where \mathfrak{h}_{σ^2} is the unique probability measure with free cumulants $\kappa_n(\mathfrak{h}_{\sigma^2}) = \frac{(\sigma^2 n)^{n-1}}{n!}$.

Proof. First, we see by Stirling's Formula that

$$\lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k, n)_{2,1}} \kappa_{Kr(\pi)}(\mu) = \lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k, n)_{2,1}} \sigma^{2n-2} = \frac{n^{n-1}}{n!} \sigma^{2n-2}. \quad (27)$$

Therefore, by Equation (11), $|NC(k, n)_{2,1}|$ and $|NC_k(n)|$ are of order k^{n-1} as $k \rightarrow \infty$ and

$$\frac{|NC_k(n)| - |NC(k, n)_{2,1}|}{|NC_k(n)|} \rightarrow 0.$$

By using the bound $|\kappa_n(\mu)| \leq 16(L^n)$, we obtain that

$$\lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k, n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \leq \lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k, n)_{2,1}}} (16L)^n = 0. \quad (28)$$

Hence

$$\begin{aligned} \lim_{k \rightarrow \infty} \kappa_n \left(D_{1/k} \left((\mu^{\boxtimes k})^{\boxplus k} \right) \right) &= \lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \kappa_n(\mu^{\boxtimes k}) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k, n)_{2,1}} \kappa_{Kr(\pi)}(\mu) \\ &\quad + \lim_{k \rightarrow \infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k, n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \\ &= \frac{n^{n-1}}{n!} \sigma^{2n-2}. \end{aligned}$$

\square

Remark 20. In view of Remark 10, some results of this section regarding the free cumulants can be also obtained for the Boolean cumulants. In particular, a second limit theorem from [11], stating that

$$\lim_{k \rightarrow \infty} D_{1/k} \left((\mu^{\boxtimes k})^{\boxplus k} \right) = \mathfrak{s}_{\sigma^2},$$

where \mathfrak{s}_{σ^2} is the unique probability measure with Boolean cumulants $b_n(\mathfrak{h}_{\sigma^2}) = \frac{(\sigma^2 n)^{n-1}}{n!}$, can be proved following the exact same lines as in Proposition 19.

A Proof of Proposition 8

Remark 21. A useful characterization of non-crossing partitions is that, for any $\pi \in NC(n)$, one can always find a block $V = \{r+1, \dots, r+s\}$ containing consecutive numbers such that if one removes this block from π , the partition $\pi \setminus V \in NC(n-s)$ remains non-crossing.

For a partition $\pi \in NC(n)$ will often write $r \sim_{\pi} s$, meaning that r, s belong to the same block of π .

Let us introduce two operations on non-crossing partitions. For $n, k \geq 1$ and $r \leq n$, we define $I_r^k : NC(n) \rightarrow NC(n+k)$, where $I_r^k(\pi)$ is obtained from π by duplicating the element in the position r , identifying the copies and inserting $k-1$ singletons between the two copies. More precisely, for $\pi \in NC(n)$, $I_r^k(\pi) \in NC(n+k)$ is the partition given by the relations:

(1) For $1 \leq m_1, m_2 \leq r$,

$$m_1 \sim_{I_r^k(\pi)} m_2 \Leftrightarrow m_1 \sim_{\pi} m_2.$$

(2) For $r+k \leq m_1, m_2 \leq n+k$,

$$m_1 \sim_{I_r^k(\pi)} m_2 \Leftrightarrow m_1 - k \sim_{\pi} m_2 - k.$$

(3) For $1 \leq m_1 \leq r$ and $r+k+1 \leq m_2 \leq n+k$,

$$m_1 \sim_{I_r^k(\pi)} m_2 \Leftrightarrow m_1 \sim_{\pi} m_2 - k.$$

(4) $r \sim_{I_r^k(\pi)} r+k$.

The operation $\tilde{I}_r^k : NC(n) \rightarrow NC(n+k)$ consists of inserting an interval block of size k between the positions $r-1$ and r in π . We will skip the explicit definition.

The importance of these operations is that they are linked by the relation

$$Kr(I_r^k(\pi)) = \tilde{I}_r^k(Kr(\pi)). \quad (29)$$

Our operations preserve properties of partitions, as shown in the following lemma.

Lemma 22. Let $\pi \in NC(nk)$, $r \leq nk$, $s \geq 1$. Then

- i) π is k -preserving if and only if $I_r^{sk}(\pi)$ is k -preserving.
- ii) π is k -completing if and only if $I_r^k(\pi)$ is k -completing.
- iii) π is k -divisible if and only if $\tilde{I}_r^{sk}(\pi)$ is k -divisible.
- iv) π is k -equal if and only if $\tilde{I}_r^k(\pi)$ is k -equal.

Proof. i) By definition of $I_r^k(\pi)$, the relations indicated by $I_r^k(\pi)$ are obtained by relations indicated by π , with possible shifts by ks (which do not modify congruences modulo k). Hence the equivalence follows.

ii) One should think of the block intervals of ρ_n^k as vertices of a graph. For $\pi \in NC(nk)$, an edge will join two vertices V, W , if there are elements $r \in V$, $s \in W$ such that $r \sim_{\pi} s$. Then $\pi \vee \rho_n^k = 1_{nk}$ if and only if the graph is connected.

It is easy to see that the effect of I_r^k on the graph of π is just splitting the vertex corresponding to the block V containing r into 2 vertices V_1, V_2 . The edges between all other vertices are preserved, while the edges which were originally joined to V will now be joined either to V_1 or V_2 . Finally, the last additional relation $r \sim_{I_r^k(\pi)} r+k$ means an edge joining V_1 to V_2 . Therefore, it is clear that the connectedness of the two graphs are equivalent.

iii) and iv) are trivial. □

Now we want to show that we can produce all partitions of our interest by applying our operations to elementary partitions.

Lemma 23. *i) Let $\pi \in NC(kn)$ be k -preserving. Then there exist $m \geq 0$ and numbers $q_0, q_1, \dots, q_m, r_1, \dots, r_m$ such that*

$$\pi = I_{r_m}^{kq_m} \circ \dots \circ I_{r_1}^{kq_1}(0_{q_0}). \quad (30)$$

ii) Let $\pi \in NC(kn)$ be k -completing. Then there exist $m \geq 0$ and numbers r_1, \dots, r_m such that

$$\pi = I_{r_m}^k \circ \dots \circ I_{r_1}^k(0_k). \quad (31)$$

iii) Let $\pi \in NC(kn)$ be k -divisible. Then there exist $m \geq 0$ and numbers $q_0, q_1, \dots, q_m, r_1, \dots, r_m$ such that

$$\pi = \tilde{I}_{r_m}^{kq_m} \circ \dots \circ \tilde{I}_{r_1}^{kq_1}(1_{q_0}). \quad (32)$$

iv) Let $\pi \in NC(kn)$ be k -equal. Then there exist $m \geq 0$ and numbers r_1, \dots, r_m such that

$$\pi = \tilde{I}_{r_m}^k \circ \dots \circ \tilde{I}_{r_1}^k(1_k). \quad (33)$$

Proof. i) We use induction on n . For $n = 1$ the only k -preserving partition is 0_k , so the statement holds. So assume that i) holds for $n \leq m$. For $\pi \in NC^k(m)$ suppose that there exist $1 \leq r < r + sk \leq km$ such that $r \sim_\pi r + sk$ and $r + 1, \dots, r + sk - 1$ are singletons of π (if no such pair (r, s) exist, necessarily $\pi = 0_{mk}$ and we are done). Then it's easy to see that $\pi = I_r^{sk}(\pi')$ for some $\pi' \in NC((n-s)k)$. By Lemma 22 i) π' is k -preserving. By induction hypothesis π' has a representation as in Equation (30) and hence, so does $\pi = I_r^{sk}(\pi')$.

The proof of ii) is similar. The proofs of iii) and iv) are trivial using Remark 21. \square

Proof of Proposition 8. We only show the first implication of i). The converse and ii) are similar.

Let $\pi \in NC(kn)$ be k -preserving. Then by Lemma 23 i) we can express it as

$$\pi = I_{r_m}^{kq_m} \circ \dots \circ I_{r_1}^{kq_1}(1_{q_0}).$$

But then we can apply Equation (29) at every step, obtaining

$$Kr(\pi) = Kr(I_{r_m}^{kq_m} \circ \dots \circ I_{r_1}^{kq_1}(0_{q_0})) \quad (34)$$

$$= \tilde{I}_{r_m}^{kq_m} \circ Kr(I_{r_{m-1}}^{kq_{m-1}} \circ \dots \circ I_{r_2}^{kq_2} \circ I_{r_1}^{kq_1}(1_{q_0})) \quad (35)$$

$$= \tilde{I}_{r_m}^{kq_m} \circ \tilde{I}_{r_{m-1}}^{kq_{m-1}} \circ Kr(I_{r_{m-2}}^{kq_{m-2}} \circ \dots \circ I_{r_2}^{kq_2} \circ I_{r_1}^{kq_1}(1_{q_0})) \quad (36)$$

$$\vdots \quad (37)$$

$$= \tilde{I}_{r_m}^{kq_m} \circ \dots \circ \tilde{I}_{r_2}^{kq_2} \circ \tilde{I}_{r_1}^{kq_1}(Kr(1_{q_0})) \quad (38)$$

$$= \tilde{I}_{r_m}^{kq_m} \circ \dots \circ \tilde{I}_{r_2}^{kq_2} \circ \tilde{I}_{r_1}^{kq_1}(0_{q_0}), \quad (39)$$

which, by Lemma 22 iii) is k -divisible. \square

References

- [1] O. Arizmendi. k -divisible elements in free probability. In preparation.
- [2] O. Arizmendi. Statistics of blocks in k -divisible non-crossing partitions. arXiv 1201.6576.
- [3] S. T. Belinschi and A. Nica, η -series and a Boolean Bercovici-Pata bijection for bounded k -tuples, *Adv. Math.* **217** (2008), 1–41.
- [4] P. H. Edelman, Chain enumeration and non-crossing partitions, *Disc. Math.* **31** (1980) 171–180.
- [5] V. Kargin. The norm of products of free random variables, *Probab. Theory Relat. Fields* **139**, (2007) 397–413.
- [6] V. Kargin. On the asymptotic growth of the support of free multiplicative convolutions *Elect. Comm. in Probab.* **13** (2008), 415–421.

- [7] T. Kemp, R. Speicher Strong Haagerup inequalities for free R-diagonal elements *J. Funct. Anal.* **251** (2007), 141–173.
- [8] G. Kreweras, Sur les partitions non croisés d'un cycle, *Discrete Math* **1**, (1972) 333–350.
- [9] B. Krawczyk, R. Speicher. Combinatorics of free cumulants, *Journal of Combinatorial Theory* Series A **90** (2000), 267–292.
- [10] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Mathematical Society Lecture Notes Series **335**, Cambridge University Press, Cambridge, 2006.
- [11] N. Sakuma and H. Yoshida. *New limit theorems related to free multiplicative convolution*. 2011. arXiv 1103.6156v1.